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ON THE DYNAMICS OF A GAS BUBBLE IN A VISCOUS INCOMPRESSIBLE LIQUID

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A. P. FROLOV
(Moscow)

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It is generally agreed that intense oscillations of cavity bubbles without collapse, constitute one of the main reasons for the cavity erosion of materials. When the dimension of a cavity bubble reaches a certain limiting value, strong pressure pulses may occur in the surrounding liquid, which can cause erosion by local cyclic loads [1 and 2].

Oscillations of cavity bubbles in a viscous liquid, exhibit a number of distinctive features caused by the viscosity. Authors of [3 and 4] noted the fundamental influence of viscosity while investigating the behavior of a spherical cavity in a viscous, incompessible liquid. The existence of two different types of motion was discovered: bubbles which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

Below we find that, when the bubble in a viscous incompressible liquid is filled with gas, then two modes of motion exist, depending on the initial radius of the bubble, oscillatory or monotonically aperiodic.

Authors of [5] use dimensional analysis to derive a qualitative formula defining the critical bubble size D_* separating the inertial and inertialess mode of expansion of a gaseous sphere in a viscous liquid

$$D_* = \left(\frac{\mu\tau_*}{\rho}\right)^{1/2}$$

where μ and ρ are the dynamic viscosity and density of the liquid, respectively, and τ_* is the characteristic time of the process, determined experimentally. Below we derive a formula for the critical diameter of the gas bubble.

Let us suppose that a spherical gas bubble is situated in an infinite, viscous, incompressible liquid. We assume that the pressure and density of the gas are uniform throughout the bubble. This of course is true, provided that the velocity of the boundary of the gaseous sphere is much smaller than the velocity of sound in the gas at a given temperature. Viscosity of gas is assumed to be negligible. The following nonlinear, second order, differential equation [5] describes the variation in the radius of the bubble

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{4\mu}{\rho R}\frac{dR}{dt} + \frac{2\sigma}{\rho R} + \frac{p_0 - p'}{\rho} = 0 \tag{1}$$

and the initial conditions are

$$R = R_0, \quad dR/dt = 0 \quad \text{when} \quad t = 0$$

Here $R = R(t)$ is the radius of the bubble, σ denotes the surface tension of the liquid, p_0 denotes constant pressure of the liquid away from the bubble and p' denotes the pressure of gas within the bubble.

Assuming that the process of expansion and contraction of gas within the bubble is adiabatic, we can derive the following relation between p' and R at any instant

$$p' = p_0' \left(\frac{R_0}{R}\right)^{3n} \tag{2}$$

when n is the adiabatic index and p'_0 is the initial pressure of gas within the bubble. In the case of adiabatic oscillations [6] of a bubble in water, we have $n = 4/3$. We also assume that there is no diffusion of gas through the boundary of the bubble.

The process of obtaining the general solution of (1) is fairly difficult even with various simplifying assumptions. Some conclusions concerning the appearance of cavitation described by this equation can, however, be drawn by perusing some general properties of the integral curves of (1) on the phase plane. This method gives a measure of success in the qualitative investigations of nonlinear differential equations.

Let us introduce the following dimensionless variables:

$$u = \frac{R}{R_0}, \quad \tau = \frac{t}{R_0}\left(\frac{p_0'}{\rho}\right)^{1/2} \tag{3}$$

Then Eq.(1) becomes

$$uu'' + \frac{3}{2}u'^2 + \frac{4\mu}{R_0\sqrt{\rho p_0'}}\frac{u'}{u} + \frac{2\sigma}{R_0 p_0'}\frac{1}{u} + \frac{p_0}{p_0'} - u^{-4} = 0 \tag{4}$$

Assuming that

$$u' = \frac{du}{d\tau} = y(u), \quad a = \frac{4\mu}{R_0\sqrt{\rho p_0'}}, \quad b = \frac{2\sigma}{R_0 p_0'}, \quad c = \frac{p_0}{p_0'} \tag{5}$$

we can reduce (4) to the following first order differential equation:

$$\frac{dy}{du} = \frac{1 - cu^4 - bu^3 - ayu^3 - 1.5y^2u^4}{yu^5} \tag{6}$$

which has one isolated singularity [7]

$$y = 0, \quad u = u_* \tag{7}$$

Here u_* is the only positive root of

$$1 - bu^3 - cu^4 = 0 \tag{8}$$

We note that u_* can be > 1, or < 1.

Near the singularity we can write Eq.(6) as

$$\frac{dy}{dx} = \frac{-(4cu_*^3 + 3bu_*^2)x - ayu_*^3 + P_2(x, y)}{yu_*^5 + Q_2(x, y)} \tag{9}$$

where $x = u - u_*$, while $P_2(x, y)$ and $Q_2(x, y)$ are functions which are second order infinitesimals, when x, y are first order infinitesimals.

Since in the present case

$$(3b + 4cu_*)u_*^7 \neq 0$$

we shall use the following characteristic equation

$$\lambda^2 + au_*\lambda + (3b + 4cu_*)u_*^3 = 0 \tag{10}$$

to determine the character of the singularity. The discriminant of (10) is equal to

$$d = (3b + 4cu_*)u_*^3 - 1/4\, a^2u_*^2 \tag{11}$$

Investigating the roots of (10) we find that two basically distinct possibilities may exist. They are:

1) roots of the characteristic equation are real, distinct and of the same sign ($d < 0$), consequently the singularity is a node. All integral curves in the neighborhood of this singularity enter it at the same angle, and the common tangent is given by

$$y + kx = 0, \qquad k = \frac{a + [a^2 - 4u_*(3b + 4cu_*)]^{1/2}}{2u_*^2}$$

As $x \to 0$, the velocity of the boundary of the bubble tends also to zero in a linear manner;

2) roots of the characteristic equation are complex conjugate. The singular point is a focus, each integral curve is contained between two logarithmic spirals and approaches the point $y = 0$, $u = u_*$, circumscribing it an infinite number of times. The gas bubble oscillates infinitely long with diminishing amplitude.

When $d = 0$, we have the intermediate case when the characteristic equation (10) has a multiple root and the singularity is a degenerate node. All integral curves enter the singularity and are tangent at this point to

$$y + \frac{ax}{2u_*^2} = 0$$

The critical bubble radius separating two distinct modes of behavior, is obtained by solving the following pair of equations

$$d = 0, \quad 1 - bu_*^2 - cu_*^4 = 0 \qquad (12)$$

If the initial radius is larger than the critical one, we have a focus, if it is smaller – we have a node. In the case of a perfect fluid ($\mu = 0$) Eq.(4) can be integrated to yield

$$y^2u^4 + \frac{2p_0'}{\rho}R_0^4 + \frac{2}{3}\frac{p_0}{\rho}u^4 + \frac{2\sigma}{\rho}u^3 + c_1u = 0$$

where c_1 is a constant of integration.

Considering this integral near the singularity and neglecting the third and higher order infinitesimals, we obtain an equation of the ellipse

$$y^2u_*^4 + u^2(3bu_* + 4cu_*^2) + c_2y + c_3u + c_4 = 0, \; (c_2, c_3, c_4 = \text{const}; \; c_4 < 0) \qquad (13)$$

with the singularity situated at its center. Here the gas bubble performs a steady oscillation.

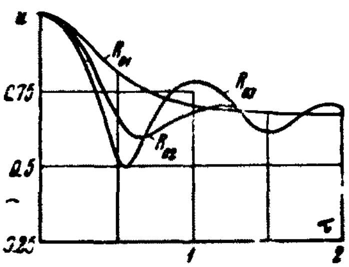

Fig. 1

It should be noted that there is a distinct difference between the process of collapsing of a bubble in a viscous liquid and that of expansion/contraction of a gaseous bubble discussed above. In the first case the surface tension has a qualitative effect on the collapse of a cavity (duration of the process remaining finite for any value of the viscosity coefficient [4]), while in the latter case two distinct modes of behavior are observed even in the absence of surface tension, the only variable being the critical radius of the bubble.

In particular, putting $b = 0$ in (12), we obtain

$$R_{0*} = \frac{\mu}{\sqrt{\rho}\,(p_0p_0')^{0.25}} \qquad (14)$$

The actual value of the critical radius is fairly small, e.g. when $p_0 = 1$ atm and $p_0' = 0.2$ atm, we have for water $R_{0*} = 0.178 \cdot 10^{-3}$ mm and for glycerine, $R_{0*} = 0.24$mm. Fig. 1 shows the results of a direct computation using Eq.(1), with $b = 0$, for the values $R_{01} = 0.15 \cdot 10^{-3}$ mm $< R_{0*}$, $R_{03} = 1 \cdot 10^{-3}$ mm $> R_{0*}$ and $R_{02} = 0.4 \cdot 10^{-3}$ mm $> R_{0*}$

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INVESTIGATION OF STABILITY OF SOLUTIONS OF SOME NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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S. K. PERSIDSKII
(Alma-Ata)
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We study a system of nonlinear differential equations (1). Imposing certain restrictions on the functions appearing on the right in (1) and on the roots of the secular equation, we obtain a number of stability theorems, on stability in the large and on the instability of the zero solution.

Let us consider the following system of differential equations

$$\frac{dx_s}{dt} = p_{s1}\varphi_1(t, x_1, \ldots, x_n) + \ldots + p_{sn}\varphi_n(t, x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{1}$$

where P_{sk} are real constants, while functions φ_s are defined and continuous in the region given by $(h)\ t \geqslant 0,\ \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} < A, \qquad (\varphi_s(t, 0, \ldots, 0) \equiv 0)$

In certain isolated cases we can base our deductions about the stability or instability of the zero solution of (1) on the properties of the roots of

$$\det \| p_{sk} - \lambda\delta_{sk} \| = 0 \tag{2}$$

For example, we can formulate the following theorems.

Theorem 1. Let the right sides of (1) be such that the function

$$u(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} a_s\varphi_s(t, x_1, \ldots, x_n) \tag{3}$$

where at least one of the numbers $a_s \neq 0$, is sign-definite. If, at the same time, Eq. (2) has no zero roots, then the zero solution of (1) is unstable.

Proof. We shall seek the following linear form

$$v(x_1, \ldots, x_n) = \sum_{s=1}^{n} b_s x_s \tag{4}$$

whose total derivative satisfies, by (1), the relation